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Streamline Diffusion Weak Galerkin Finite Element Methods for Linear Unsteady State Convection Diffusion Equations and Error Analysis

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Abstract

In this paper, the streamline diffusion weak Galerkin finite element method is proposed and analyzed for solving unsteady time convection diffusion problem in two dimension. The v-elliptic property and the stability of this scheme are proved in terms of some conditions. We derive an error estimate in $L^2(\mu)$ and $H^1(\mu)$ norm. Numerical experiments have demonstrated the effectiveness of the method in solving convection propagation problems, and the theoretical analysis has been validated.

Keywords: streamline diffusion; weak Galerkin finite element; discrete gradient; stability; error analysis.

1 Introduction

This paper is dealing with problems that are mainly of a hyperbolic character, such as the convection-diffusion problem with a small or vanishing diffusion. This type of problem is typically encountered in fluid mechanics, gas dynamics, or the propagation of waves [16].

Let us consider the following example of a linearly unstable convection-diffusion problem. We are seeking the unknown function $\omega = \omega(x;t)$ which satisfies

$$\begin{aligned} &\omega_t - \epsilon\Delta\omega + \beta\cdot\nabla\omega + c\omega = f(x,t), \quad (x,t)\in\mu\times(0,T], \\ &\omega(x,0) = 0, \quad x\in\mu, \\ &\omega(x,t)\mid_{\Gamma} = g, \quad (x,t)\in\Gamma\times(0,T]. \end{aligned} \tag{1}$$

Consider a region $\mu \subset R^2$ with a boundary Γ that is Lipschitz continuous. The function $\omega = \omega(x,t)$ is represented by its gradient $\nabla\omega$. Furthermore, the symbol $\epsilon > 0$ represents the diffusion coefficient, β represents the convection coefficient, and f and g are given functions. The phenomena of convection, diffusion and interaction are widespread in various scientific and engineering fields. These processes are crucial in fluid dynamics, heat and mass transfer, hydrology, and related fields. Convection-dominated diffusion equations, such as fluid flow and semiconductor device modelling, are used in practical physics. Solutions to these problems typically exhibit layers and narrow regions where some derivatives of the first-order solution are significantly large. Finding accurate solutions within these layers is a significant challenge as these are areas where the solution is still ongoing, but the gradients are substantial. Layers cannot be resolved because their width is generally much less than the width of the available digital grid.

Solving the linear convective diffusion equation can be challenging under singular perturbations despite its seemingly straightforward nature. Thin layers, characterized by rapid changes in the solution and its derivatives, often emerge in the boundary value problem due to the extremely low diffusion coefficient [25]. Finding analytical solutions for mathematical models encapsulating convective and diffusive effects is generally considered impractical. Therefore, numerical methods are often necessary to estimate unknown parameters. These equations' scrutiny and computational resolution have attracted considerable attention, reflecting the inherent complexity of concurrently addressing convective and diffusive processes. However, traditional numerical methods could be more effective in typical cases where convection dominates over diffusion, as the approximate solutions often contain unwanted oscillations. This is related to the fact that solutions to convection-dominated problems often exhibit layers, confined regions characterized by sudden changes in the solution. This aspect attracts several specialists in various fields [24].

Consequently, many individuals from various fields, including athletes and engineers, have shown interest in this approach. The physical process of diffusion imparts a property through the movement of liquid molecules, while convection involves the motion of a flow that conveys specific attributes. Partial differential equations describe the behaviour of fluids undergoing mass transfer, forced heat, or vortex transfer. Conventional numerical methods may not provide reliable approximations in such cases. Studying the effective and computational methods for the problem of computational mesh not being on the same scale as the layers holds practical significance and has captured the attention of many researchers. When dealing with complex domains or layered structures, fitted operator techniques are frequently used, predominantly employing upwind-type schemes. Upwinding aims to introduce artificial diffusion/viscosity to counteract convection and maintain the stability of a traditional discretization approach. Upwind schemes were initially introduced in finite difference methods and later extended to encompass finite element methods. Several stabilization methods have been presented to avoid these spurious oscillations, including streamline-upwind Petrov-Galerkin (SUPG) methods [6, 14], local projection stabilization methods [3, 23, 25], discontinuous Galerkin methods [2, 26, 18], artificial diffusivity methods [9, 19, 20], upwinding techniques [27, 17, 5], streamline diffusion methods [29, 13], Petrov-Galerkin approaches [8, 30, 21] and several alternatives were compared in [4, 15]. Although much work has been devoted to the numerical solution of convection-dominated equations, no method could still be considered entirely satisfactory. For example, the popular SUPG method suffers from producing overshoots and undershoots near the layer region.

The weak Galerkin finite element method (WGFEM) emerges as an innovative framework for tackling partial differential equations. Initially developed by Wang and Ye [28, 22], WGFEM proves effective in numerically solving second-order elliptic equations. Notably, this numerical technique allows for the utilization of discontinuous functions and features a straightforward formulation independent of parameters, courtesy of weak function and weak gradient concepts. Gao et al. [10] presented numerical methods for Sobolev problems using the weak Galerkin finite element method. Zhang and Lin [31] explored the same method for steady-state Navier-Stokes equations. Applying a similar approach, Cheichan et al. [7] resolved the one- and two-dimensional nonlinear convection-diffusion equations. Hussein and Kashkool [11, 12] employed the weak Galerkin technique to solve interconnected one-dimensional and two-dimensional Burgers equations.

This project will modify the weak Galerkin finite element method to include a streamlined diffusion term. This additional term is written as $-\delta u_{\beta\beta}$, which introduces diffusion only in the direction of β. Moreover, we will study the semi-discrete streamline diffusion weak Galerkin finite element method for linear unsteady state convection-diffusion problems. We present the elliptic property and the stabilization for semi-discrete schemes. In the L^2 and H^1 norm, we will analyze error estimates and a numerical experiment.

2 Outline and Preliminaries

These Sobolev spaces are defined according to a standard set of practices $H^C(Q)$ and its related inerts $(\cdot,\cdot)_{C,Q}$, norms $\|\cdot\|_{C,Q}$, and seminorms $|\cdot|_{C,Q}$ for $C \geq 0$. For example, [28] for any integer $C \geq 0$, the seminorm $|\cdot|_{C,Q}$ is given by

$$|v|_{C,Q} = \left(\sum_{|n|=C} \int_Q |\partial^n v|^2 \, dQ \right)^{\frac{1}{2}},$$

with the usual notation

$$n = (n_1, n_2), \quad |n| = n_1 + n_2, \quad \partial^n = \partial_{x_1}^{n_1} \partial_{x_2}^{n_2}.$$

The Sobolev norm $\|\cdot\|_{s,Q}$ is given by

$$\|v\|_{s,Q} = \left(\sum_{i=0}^{Q} |v|_{i,s}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(Q)$ coincides with $L^2(Q)$, for which the norm and the internal results are indicated by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, respectively. When $Q = \mu$, we are dropping the subscript Q in the norm and the internal notation used for products. The space $H(\mathrm{div}; \mu)$ is the collection of vector-valued functions on the μ, which are integrable as well as their diverging squares, which means that

$$H(\text{div};\mu) = \left\{ \mathbf{v} : \mathbf{v} \in \left[L^2(\mu)\right]^2, \nabla \cdot \mathbf{v} \in L^2(\mu) \right\}.$$

The norm in $H(\text{div};\mu)$ is defined by

$$\|\mathbf{v}\|_{H(\text{div};\mu)} = \left(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

The standard Galerkin finite element method (weak form) of Equation (1) seeks $\omega \in H_0^1(\mu)$ such that $\omega = g$ on $\Gamma \times (0,T]$ and

$$(\omega_t, v) + (\epsilon \nabla \omega, \nabla v) + (\beta \cdot \nabla \omega, v) + (c\omega, v) = (f, v), \forall v \in H_0^1(\mu). \tag{2}$$

3 Weak Galerkin Finite Element Method

The approach of a gradient operator with a weak definition for discontinuous functions was originally used in [28] to describe weak Galerkin finite elements. The weak gradient operator is used in weak Galerkin finite element methods to approximate weak gradients and divergences. Let K, which represents a polygonal domain with an interior and a boundary. The function ω is a vector function, where K^0 represents the interior of the domain and K^b represents the boundary. For better understanding, technical terms are explained in detail. $W(K)$ refers to the features of the function space K that are characterized by their weakness,

$$W(K) = \left\{ \omega = \left\{\omega^0, \omega^b\right\} : \omega^0 \in L^2(K^0), \omega^b \in H^{\frac{1}{2}}(\partial K) \right\}.$$

Definition 3.1. *The dual of $L^2(K)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $\omega \in W(K)$, the weak gradient of ω is defined as a linear functional $\nabla_d \omega$ in the dual space of $H(\text{div}, K)$ whose action on each $q \in H(\text{div}, K)$ is given by*

$$(\nabla_d \omega, q) := -\int_K \omega^0 \nabla \cdot q dK + \int_{\partial K} \omega^b q \cdot \mathbf{n} ds,$$

where $\mathbf{n}$ is the outward normal direction to ∂K.

This part will introduce two crucial finite element spaces for defining numerical schemes. The triangular partition N_h is a subdivision of the domain μ into elements $\aleph$, where each element has a uniform mesh size of h. $P_m(\aleph)$ refers to the collection of polynomials defined on the internal of $\aleph$ with a degree greater than m. Moreover, we denote the collection of polynomials on the boundary of $\aleph$ with a degree less than or equal to l as $P_l(\partial\aleph)$. Let us define a discrete weak function ω as $\omega = \left\{\omega^0, \omega^b\right\}$, where ω^0 belongs to the space $P_m(\aleph)$ and ω^b belongs to the space $P_l(\partial\aleph)$. The variables m and l denote numbers greater than or equal to zero. The specified region is often written as

$$W(\aleph, m, l) = \left\{\omega = \left\{\omega^0, \omega^b\right\} : \omega \in P_m\left(\aleph^0\right), \omega^b \in P_l(\partial\aleph)\right\}.$$

The finite element space for this would be constructed by combining the space $W(\aleph, m, l)$ for each triangle $\aleph$ in the triangulation N_h. The weak finite element space can be precisely defined as

$$\mathcal{S}_h(m,l) = \left\{\omega = \left\{\omega^0, \omega^b\right\} : \left.\left\{\omega^0, \omega^b\right\}\right|_T \in W(\aleph, m, l), \forall \aleph \in N_h\right\}.$$

The associated finite element space is the union of spaces called $W(\aleph, m, l)$ for all triangles $\aleph$ in triangulation N_h. The weak finite element space can be denoted as

$$\mathcal{S}_h^0(m,l) = \left\{\omega = \left\{\omega^0, \omega^b\right\} \in \mathcal{S}_h(j,l) : \omega^b|_{\partial\aleph\cap\partial\mu} = 0, \quad \forall\aleph \in N_h\right\}.$$

We will define a discrete weak gradient operator by explicitly describing the operator ∇_d within a polynomial subspace of the space $H(\mathrm{div}, K)$. Let r be a non-negative integer greater than or equal to zero to achieve this goal. The set of polynomials on $\aleph$ whose degree does not exceed r is called $P_r(\aleph)$. Let us examine the subspace $V(\aleph, r)$, which is a subset of the space $[P_r(\aleph)]^2$ and consists of vector-valued polynomials of degree r. The discrete weak gradient operator of the function ω on each element $\aleph$, denoted as $\nabla_{d,r}$ for each $\omega = \left\{\omega^0, \omega^b\right\} \in \mathcal{S}_h(m,l)$, can be represented by the following equation [28]:

$$\int_\aleph \nabla_{d,r}\omega \cdot q d\aleph = -\int_\aleph \omega^0 \nabla \cdot q d\aleph + \int_{\partial\aleph} \omega^b q \cdot n ds, \forall q \in V(\aleph, r).$$

We determine the bilinear form that follows $\omega, v \in S_h(m, \ell)$,

$$a_{WG}(\omega, v) = (\epsilon\nabla_{d,r}\omega, \nabla_{d,r}v) + \left(\beta \cdot \nabla_{d,r}\omega, v^0\right) + \left(c\omega^0, v^0\right),$$

where

$$\begin{aligned}
(\epsilon\nabla_{d,r}\omega, \nabla_{d,r}v) &= \int_\mu \epsilon\nabla_{d,r}\omega \cdot \nabla_{d,r}v d\mu,\\
\left(\beta \cdot \nabla_{d,r}\omega, v^0\right) &= \int_\mu \beta \cdot \nabla_{d,r}\omega^0 v d\mu,\\
\left(c\omega^0, v^0\right) &= \int_\mu c\omega^0 v^0 d\mu.
\end{aligned}$$

A numerical approximation for equation can be obtained by searching for the values of $\omega_h = \left\{\omega^0, \omega^b\right\}$ in the function space $S_h(m,l)$ that satisfy the boundary condition $\omega^b = Q_b g$ on the boundary $\partial\mu$, together with the given equation,

$$\left((\omega_h)_t, v^0\right) + a_{WG}(\omega_h, v) = \left(f, v^0\right), \quad \forall v = \left\{v^0, v^b\right\} \in S_h^0(m,l). \tag{3}$$

The expression $Q_b g$ approximates the boundary value in the polynomial space $P_l(\partial\aleph \cap \partial\mu)$. For simplicity, we will use $Q_b g$ as the default L^2 projection for each boundary segment, ignoring other approximations of the boundary value $\omega = g$. Let us denote $\omega \in H^{k+1}(\mu)$, where $k \geq 1$, as the exact solution of Equation (1). To continue, we will present the L^2 projections,

$$\begin{aligned}
&Q_h : L^2(\aleph) \to P_k(\aleph); \forall\aleph \in \aleph_h,\\
&R_h : \left[L^2(\aleph)\right]^2 \to \left[P_{k-1}(\aleph)\right]^2; \forall\aleph \in \aleph_h.
\end{aligned}$$

Two beneficial identities can be easily observed to exist

$$\nabla_{d,r}(Q_h u) = R_h(\nabla u), \quad \forall u \in H^1(\aleph).$$

This method will yield satisfactory results if the magnitude of ϵ is significantly greater than that of $\|\beta\|_{L_\infty(\mu)} h$. However, this approach may produce a solution that exhibits oscillations and deviates significantly from the exact solution if the magnitude of ϵ is significantly smaller than that of $\|\beta\|_{L_\infty(\mu)} h$. Furthermore, as shown in Figure 1, precision is lacking in approximating the exact solution over the entire range. The lack of smoothness in the exact solution of many interesting elliptic equations is the reason for the importance of this topic. Recently, there have been improvements in dealing with these difficulties, allowing the creation of modified unconventional WGFEM that exhibit favorable convergence characteristics when used in elliptic cases.

We solved the problem by using a method known as the classical artificial diffusion of weak Galerkin finite element method (CADWGFEM). Although this technique provides non-oscillatory solutions, it has the disadvantage of introducing a significant amount of additional diffusion, which comes at a significant cost in terms of increased diffusion. Specifically, this method introduces a diffusion term $-h\omega_{\beta\beta}$ in the β direction, perpendicular to the streamline, which accounts for crosswind diffusion [1]. As a result, abrupt fronts or jumps across a streamline are exceptionally damped. Furthermore, an additional term $-\delta\Delta\omega$ is included. It is important to note that this method only achieves first-order accuracy, and even for smooth solutions, the error is at best in the order of $O(h)$. The resulting solution is a numerical approximation. It is characterized by smoothing effects on the transformations, similar to the solution derived from the fundamental equation.

4 Streamline Diffusion Weak Galerkin Finite Element Method

Several specific techniques have been devised for the numerical resolution of convective propagation problems. To study the numerical solution, we use a uniform mesh with a diameter of h and N_h is quasi-uniform or $\|\beta\|$ it is equal to 1. Traditional numerical methods show suboptimal performance when the disturbance parameter ϵ is smaller than the h. In such cases, the generated solutions often exhibit unwanted spurious oscillations throughout the entire domain. To address the challenges associated with this excessive diffusion, recent research has explored alternative techniques, such as streamline diffusion [16] and streamline upwind Petrov-Galerkin (SUPG) methods [14]. These newer approaches aim to maintain stability while minimising unnecessary crosswind smoothing. The crosswind motion aligned parallel to the streamlines exhibits a smoother and uninterrupted transition in these refined methods, providing a more balanced trade-off between stability and solution accuracy.

Several specific techniques have been devised for the numerical solution of convective propagation problems. We will leverage one of the most widely used techniques, the WGFEM. The differential equation is modified to include a streamline diffusion term that introduces diffusion only in the direction β of the streamlines (often called the streamline direction for static problems). The stabilization is realized by adding the term $-\delta u_{\beta\beta}$ into the equation, with δ being a positive parameter. This supplementary diffusion aims to yield a more stable solution true to physical realities by diminishing oscillations. Specifically, δ is set equal to $h-\varepsilon$ when $\varepsilon < h$, and δ is zero if $\varepsilon \geq h$. This approach is the streamline diffusion weak Galerkin finite element method (SLDWGFEM). The SLDWGFEM is used to solve (1), which reads: Find the solution $\omega_h = \{\omega^0, \omega^b\} \in S_h(m,l)$ that satisfies the boundary condition $u^b = Q^b g$ on $\partial\omega$:

$$\left((\omega_h)_t\, , v^0\right) + a_{SLDWG}\left(\omega_h, v\right) = \left(f, v^0\right), \quad \forall v = \left\{v^0, v^b\right\} \in S_h^0(m,l), \tag{4}$$

where

$$\begin{aligned} a_{SLDWG}\left(\omega_h, v\right) = &\left(\epsilon \nabla_{d,r}\omega_h, \nabla_{d,r} v\right) + \delta\left(\beta\cdot\nabla_{d,r}\omega_h, \beta\cdot\nabla_{d,r} v\right) \\ &+ \left(\beta\cdot\nabla_{d,r}\omega_h, v\right) + \left(c\omega_h, v^0\right). \end{aligned} \tag{5}$$

This method introduces less crosswind diffusion than the CADWGFEM. For (4), we will now demonstrate the elliptic property of SLDWGFE.

Lemma 4.1. *The bilinear form $a_{SLDWG}\left(\omega_h, v\right)$ described in (5) on the weak finite element space $S_h(m,l)$ is provided by: There exists a positive constant φ that satisfies*

$$a_{SLDWG}\left(v_h, v_h\right) \geq \varphi\left(\|\nabla_{d,r} v\|^2 + \left\|v^0\right\|^2\right),$$

for all $v_h \in S_h(j,i)$.

Proof. Substituting $\omega = v$ into (5), we get

$$a_{SLDWG}(v_h, v_h) = \delta(\beta.\nabla_{d,r}v, \beta.\nabla_{d,r}v) + \varepsilon(\nabla_{d,r}v, \nabla_{d,r}v) + (\beta\cdot\nabla_{d,r}v, v^0) + (v^0, v^0). \tag{6}$$

Let $B = \|\beta\|_{L_\infty(\Omega)}$ and $C = \|c\|_{L_\infty(\Omega)}$ be the L_∞-norm of the coefficients β and c, respectively. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\delta(\beta.\nabla_{d,r}v, \beta.\nabla_{d,r}v)| &\le \delta\,\|\beta\cdot\nabla_{d,r}v\|\,\|\beta\cdot\nabla_{d,r}v\| \\ &\le \delta\|\beta\|^2_{L_\infty(\Omega)}\,\|\nabla_{d,r}v\|^2 \\ &\le \delta B^2\,\|\nabla_{d,r}v\|^2, \end{aligned} \tag{7}$$

and $\left|(\beta\cdot\nabla_{d,r}v, v^0)\right| \le \|\beta\cdot\nabla_{d,r}v\|\,\|v^0\|$. Next, using ε Young's inequality, we derive

$$\begin{aligned} \|\beta\cdot\nabla_{d,r}v\|\,\|v^0\| &\le \frac{1}{2\gamma}\|\beta\cdot\nabla_{d,r}v\|^2 + \frac{\gamma}{2}\|v^0\|^2 \\ &\le \frac{B^2}{2\gamma}\|\nabla_{d,r}v\|^2 + \frac{\gamma}{2}\|v^0\|^2, \end{aligned} \tag{8}$$

and

$$\left|(cv^0, v^0) \le\right|\,\|c\|_{L_\infty(\Omega)}\,\|v^0\|^2 \le C\,\|v^0\|^2. \tag{9}$$

Substituting (7), (8), and (9) into (6), we obtain

$$\begin{aligned} a_{SLDWG}(v_h, v_h) &\ge \varepsilon\|\nabla_{d,r}v\|^2 - \left(\frac{B^2}{2\gamma}\|\nabla_{d,r}v\|^2 + \frac{\gamma}{2}\|v^0\|^2\right) + \delta B^2\|\nabla_{d,r}v\|^2 + C\|v^0\|^2 \\ &\ge \left(\varepsilon - \frac{B^2}{2\gamma} + \delta B^2\right)\|\nabla_{d,r}v\|^2 + \left(C - \frac{\gamma}{2}\right)\|v^0\|^2. \end{aligned} \tag{10}$$

In (10), the term $\left(\varepsilon - \frac{B^2}{2\gamma} + \delta B^2\right) > 0$ for suffficientely small ε reduce to $B^2\left(\delta - \frac{1}{2\gamma}\right)$, then we have

$$a_{SLDWG}(v_h, v_h) \ge \varphi\left(\|\nabla_{d,r}v\|^2 + \|v^0\|^2\right),$$

where $\varphi = \min\left\{B^2\left(\delta - \frac{1}{2\gamma}\right), \left(C - \frac{\gamma}{2}\right)\right\}$. □

5 Stability for SLDWGFEM

Since stability is intricately linked to numerical accuracy, an unstable numerical scheme can escalate errors within the simulation, which may deviate significantly from the exact solution. Thus, ensuring the stability of the SLDWGFEM is pivotal to maintaining the accuracy of simulations across temporal or spatial domains. Hence, this chapter is devoted to demonstrating the stability of the SLDWGFEM for (4).

Theorem 5.1. *For any solution ω_h to the problem (1) under the SLDWGFEM formulation in (4), let Lemma 4.1 be satisfied in $S_h(m,l)$. Given that there is a positive constant J such that the energy of the solution is bounded over time, we obtain*

$$\|\omega_h(t)\|^2 \leq \left\|\omega^0\right\|^2 + \int_0^t \frac{1}{J}(\|f(\tau)\|^2 d\tau - \int_0^t \|\nabla_{d,r}\omega_h(\tau)\|^2 d\tau + \int_0^t J\|\omega_h(\tau)\|^2 d\tau - \int_0^t \left\|\omega^0(\tau)\right\|^2 d\tau.$$

Proof. Choose $v = \omega_h$ in (4), we obtain

$$(\omega_{h,t}, \omega_h) + a_{SLDWG}(\omega_h, \omega_h) = (f, \omega_h). \tag{11}$$

From Lemma 4.1, we have

$$a_{SLDWG}(\omega_h, \omega_h) \geq \varphi\left(\|\nabla_{d,r}\omega_h\|^2 + \left\|\omega^0\right\|^2\right), \tag{12}$$

and

$$(\omega_{h,t}, \omega_h) = \frac{1}{2}\frac{d}{dt}\|\omega_h(t)\|^2. \tag{13}$$

The following can be obtained by applying the Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}(f, \omega_h) &\leq \frac{1}{\sqrt{J}}\|f\|\sqrt{J}\,\|\omega_h\| \\ &\leq \frac{1}{2J}\|f\|^2 + \frac{J}{2}\|\omega_h\|^2.\end{aligned} \tag{14}$$

Substituting (12), (13), and (14) into (11), and let $\varphi = \frac{1}{2}$, then we get

$$\frac{1}{2}\frac{d}{dt}\|\omega_h(t)\|^2 + \varphi\left(\|\nabla_{d,r}\omega_h\|^2 + \left\|\omega^0\right\|^2\right) \leq \frac{1}{2J}\|f\|^2 + \frac{J}{2}\|\omega_h\|^2. \tag{15}$$

Multiply (15) by 2 yields

$$\frac{d}{dt}\|\omega_h(t)\|^2 + \left(\|\nabla_{d,r}\omega_h\|^2 + \left\|\omega^0\right\|^2\right) \leq \frac{1}{J}\|f\|^2 + J\|\omega_h\|^2, \tag{16}$$

and integral both side from 0 to t, and applying the initial condition $\omega_h(0) = \omega^0$, we obtain

$$\|\omega_h(t)\|^2 \leq \left\|\omega^0\right\|^2 + \int_0^t \frac{1}{J}(\|f(\tau)\|^2 d\tau - \int_0^t \|\nabla_{d,r}\omega_h(\tau)\|^2 d\tau + \int_0^t J\|\omega_h(\tau)\|^2 d\tau - \int_0^t \left\|\omega^0(\tau)\right\|^2 d\tau. \tag{17}$$

□

6 The Error Analysis of SLDWGFEM in L^2 Norm

In this part, we obtain an error estimate for the SLDWGFEM. First, we examine the error equations for the SLDWGFEM approximations, denoted as ω_h, and the L^2 projection of the exact solution, denoted as ω, into the weak finite element space, denoted as $S_h(m,l)$. The L^2 projection is denoted by the symbol $Q_h\omega \equiv \left\{Q_h\omega^0, Q_h\omega^b\right\}$. It is the local L^2 projection of each triangular element $T \in N_h$ onto $P_m(T)$, where Q_h^0 is the projection inside the element and Q_h^b is the projection onto the boundary of T.

Lemma 6.1. [28] *For $\omega \in H^{1+r}(\Omega)$ with $r > 0$, we have*

$$\|\pi_h \omega - Q_h \omega\| \leq C h^{r+1} \|\omega\|_{1+r},$$
$$\|\omega - Q_h \omega\| \leq C h^{r} \|\omega\|_{1+r}.$$

Lemma 6.2. [28] *For $\omega \in H^{1+r}(\Omega)$ with $r > 0$, we have*

$$\|\pi_h(\epsilon \nabla \omega) - \epsilon \nabla \left(Q_h \omega\right)\| \leq C h^{r} \|\omega\|_{1+r}. \tag{18}$$

Lemma 6.3. *Define $e_h(t)$ as the difference between $Q_h\omega(t)$ and $\omega_h(t)$, where $e_h(t)$ belongs to the space $S_h^0(m,l)$. The time t is fixed and lies in the interval $(0,T]$. For each $v \in S_h^0(m,l)$, the error equation for SLDWGFM is given by*

$$(\partial_t e_h, v_0) + a_{SLDWG}\left(e_h, v\right) = \ell(\omega, v), \tag{19}$$

where

$$\begin{aligned}\ell(\omega, v) = {} & \delta\left(\pi_h\left(\beta \cdot \nabla_{d,r}\omega\right) - \beta \cdot R_h(\nabla \omega), \beta \cdot \nabla_{d,r} v\right) + \left(\pi_h\left(\epsilon \nabla_{d,r}\omega\right) - \epsilon R_h(\nabla\omega), \nabla_{d,r} v\right) \\ & - \left(\pi_h \omega - \left(Q^0 \omega\right), \nabla_{d,r} \cdot (\beta v)\right) + \left(c\omega - c\left(Q^0\omega\right), v^0\right).\end{aligned}$$

Proof. Consider $v = \left\{v^0, v^b\right\}$ as the testing function, where v belongs to the set $S_h^0(m,l)$. By testing (1) against v^0, together with (4), we have

$$\begin{aligned}&\left(\omega_t, v^0\right) + \left(\pi_h\left(\epsilon \nabla_{d,r}\omega\right), \nabla_{d,r} v\right) + \delta\left(\pi_h\left(\beta \cdot \nabla_{d,r}\omega\right), \beta \cdot \nabla_{d,r} v\right) \\ &\qquad - \left(\pi_h \omega, \nabla_{d,r} \cdot (\beta v)\right) + \left(c\omega_h, v^0\right) = \left(f, v^0\right).\end{aligned} \tag{20}$$

Adding and subtracting the term $a_{SLDWG}\left(Q_h\omega, v\right)$, where

$$\begin{aligned}a_{SLDWG}\left(Q_h \omega, v\right) = {} & \left(\epsilon \nabla_{d,r}\left(Q_h\omega\right), \nabla_{d,r} v\right) + \delta\left(\nabla_{d,r}\left(\beta \cdot Q_h \omega\right), \beta \cdot \nabla_{d,r} v\right) \\ & - \left(\left(Q^0\omega\right), \nabla_{d,r} \cdot (\beta v)\right) + \left(c\left(Q^0\omega\right), v^0\right).\end{aligned}$$

On the left hand side of (20) and using $\left(Q_h\omega_t, v^0\right) = \left(\omega_t, v^0\right)$, we obtain

$$\begin{aligned}&(Q_h\omega_t, v^0) + (\pi_h(\epsilon\nabla_{d,r}\omega) - \epsilon\nabla_{d,r}(Q_h\omega), \nabla_{d,r}v) + \delta(\pi_h(\beta\cdot\nabla_{d,r}\omega) - \beta\cdot\nabla_{d,r}(Q_h\omega), \beta\cdot\nabla_{d,r}v) \\ &- (\pi_h(\omega) - (Q^0\omega), \nabla_{d,r}\cdot(\beta v)) + (c\omega - c(Q^0\omega), v^0) + (\epsilon\nabla_{d,r}(Q_h\omega), \nabla_{d,r}v) \\ &+ \delta(\beta\cdot\nabla_{d,r}(Q_h\omega), \beta\cdot\nabla_{d,r}v) - ((Q^0\omega), \nabla_{d,r}\cdot(\beta v)) + (c(Q^0\omega), v^0) \\ &= (f, v^0),\end{aligned}$$

and then using $R_h(\nabla\omega) = \nabla_{d,r}\left(Q_h\omega\right)$ for $\omega \in H^1$, we obtain

$$\begin{aligned}&\left(\omega_{h,t}, v^0\right) + a_{SLDWG}\left(\omega_h, v\right) \\ &= a_{SLDWG}\left(Q_h\omega, v\right) + \left(Q_h\omega_t, v^0\right) + \left(\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon R_h(\nabla\omega), \nabla_{d,r}v\right) \\ &\quad + \delta\left(\pi_h\left(\beta.\nabla_{d,r}\omega\right) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}v\right) + \left(c\omega - c\left(Q^0\omega\right), v^0\right) \\ &\quad - \left(\pi_h(\omega) - \beta\left(Q^0\omega\right), \nabla_{d,r}\cdot(\beta v)\right),\end{aligned} \tag{21}$$

which can be restated

$$\begin{aligned}&\left(\left(\omega_h - Q_h\omega\right)_t, \ v^0\right) + a_{SLDWG}\left(\omega_h - Q_h\omega, v\right) \\ &= \delta\left(\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}v\right) \\ &\quad + \left(\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon R_h(\nabla\omega), \nabla_{d,r}v\right) - \left(\pi_h\omega - \left(Q^0\omega\right), \nabla_{d,r}\cdot(\beta v)\right) \\ &\quad + \left(c\omega_h - c\left(Q^0\omega\right), v^0\right).\end{aligned} \tag{22}$$

Equation (22) is called the error equation for SLDWGFEM. □

Lemma 6.4. *Suppose the dual of problem (1) exhibits H^{1+k} regularity. Consider $\omega \in H^{1+k}(\mu)$ as the solution to (1), with ω_h representing the SLDWGFEM approximation derived from (4) through the employment of the weak finite element spaces $S_h(m,l)$. Let $e = \omega_h - Q_h\omega$, where $Q_h\omega$ denote the L^2 projection of ω onto the corresponding finite element space, which is defined locally. Then, there exists a constant C such that*

$$\|e(t)\|^2 + C_1 \int_0^t \|e(\tau)\|^2 d\tau \leq Ch^{2k+2} \int_0^t \|\omega(\tau)\|_{1+k}^2 d\tau + \|e(0)\|^2.$$

Proof. For $t \in (0,T]$, let $e = \{\omega_h - Q_h\omega\} = \{e^0, e^b\} = \{\omega^0 - Q^0\omega, \omega^b - Q^b\omega\}$, and taking $v = e$ in (22), we have

$$\begin{aligned}(e_t, e) + a_{SLDWG}(e,e) &= (\pi_h(\epsilon\nabla_{d,r}\omega) - \epsilon R_h(\nabla\omega), \nabla_{d,r}e) \\ &\quad + \delta(\pi_h(\beta\cdot\nabla_{d,r}\omega) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}e) \\ &\quad - \left(\pi_h(\omega) - \left(Q^0\omega\right), \nabla_{d,r}\cdot(\beta e)\right) + \left(c\omega - c\left(Q^0\omega\right), e\right).\end{aligned} \tag{23}$$

The assumed H^{1+k} regularity for dual problem implies the existence of a constant C such that

$$\|\omega\|_{1+k} \leq C\|e_0\|.$$

By Cauchy-Schwartz and the Lemma 4.1 and Young's-inequality of the bilinear form, we have

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \frac{\vartheta}{2}\|e\|^2 + \frac{\vartheta}{2}\|\nabla_{d,r}e\|^2 = \sum_{i=1}^{4} \mathrm{I}^i, \tag{24}$$

where

$$\begin{aligned}\mathrm{I}^1 &= (\pi_h(\epsilon\nabla_{d,r}\omega) - \epsilon Q_h(\nabla\omega), \nabla_{d,r}e), \\ \mathrm{I}^2 &= \delta(\pi_h(\beta\cdot\nabla_{d,r}v) - \beta\cdot Q_h(\nabla\omega), \beta\cdot\nabla_{d,r}e), \\ \mathrm{I}^3 &= (\pi_h(v) - Q_h(\nabla\omega), \nabla_{d,r}(\beta e)), \\ \mathrm{I}^4 &= \left(c\omega - c\left(Q^0\omega\right), e\right).\end{aligned}$$

To estimate I^1 by Cauchy-Schwartz and Young's-inequality, we obtain

$$\begin{aligned}\left|\mathrm{I}^1\right| &\leq \frac{1}{2\emptyset}\|\pi_h(\epsilon\nabla_{d,r}\omega) - \epsilon Q_h(\nabla\omega)\|^2 + \frac{\emptyset}{2}\|\nabla_{d,r}e\|^2 \\ &\leq \frac{1}{2\emptyset}\|\pi_h(\epsilon\nabla_{d,r}\omega) - \epsilon Q_h(\nabla\omega)\|^2 + C\|e\|^2.\end{aligned}$$

By Lemma 6.2, we have

$$\left|\mathrm{I}^1\right| \leq Ch^{2k}\|\omega\|_{1+k}^2 + C\|e\|^2. \tag{25}$$

To estimate I^2 and $\delta = \mathrm{h} - \epsilon$, we have

$$\begin{aligned}\mathrm{I}^2 &= (h-\epsilon)(\pi_h(\beta\cdot\nabla_{d,r}\omega) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}e) \\ &= h(\pi_h(\beta\cdot\nabla_{d,r}v) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}e) - \epsilon(\pi_h(\beta\cdot\nabla_{d,r}\omega) - \beta\cdot R_h(\nabla\omega), \beta\cdot\nabla_{d,r}e),\end{aligned}$$

where $\mathrm{I}^2 = \mathrm{I}_1^2 - \mathrm{I}_2^2$. To find I_1^2 by Cauchy-Schwartz and Young's-inequality and Lemma 6.2, we obtain

$$\begin{aligned}\mathrm{I}_1^2 &\leq \frac{h^2}{2\emptyset}\|\pi_h(\beta\cdot\nabla_{d,r}\omega) - \beta\cdot R_h(\nabla\omega)\|^2 + \frac{\emptyset}{2}\|\beta\cdot\nabla_{d,r}e\|^2 \\ &\leq \frac{h^2\|\beta\|^2}{2\emptyset}\|\pi_h(\nabla_{d,r}\omega) - R_h(\nabla\omega)\|^2 + \frac{\emptyset\|\beta\|^2}{2}\|\nabla_{d,r}e\|^2 \\ &\leq \mathrm{Ch}^{2k+2}\|\omega\|_{1+k}^2 + \frac{\alpha}{2}\|\nabla_{d,r}e\|^2.\end{aligned} \tag{26}$$

To find I_2^2 by Cauchy-Schwartz and Young's-inequality, we obtain

$$\begin{aligned} \mathrm{I}_2^2 &= \delta\left(\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right)-\beta\cdot R_h(\nabla\omega),\beta\cdot\nabla_{d,r}e\right), \\ \left|\mathrm{I}_2^2\right| &\leq \frac{1}{2\emptyset}\left\|\pi_h\left(\epsilon\beta\nabla_{d,r}\omega\right)-\epsilon\beta\cdot R_h(\nabla\omega)\right\|^2+\frac{\emptyset}{2}\left\|\beta\cdot\nabla_{d,r}e\right\|^2. \end{aligned} \tag{27}$$

By Lemma 6.2, we have

$$\left|\mathrm{I}_2^2\right| \leq \frac{c}{2\emptyset}\|\beta\|^2 h^{2k}\|\omega\|_{1+k}^2+\frac{\emptyset}{2}\|\beta\|^2\left\|\nabla_{d,r}e\right\|^2 \leq \mathrm{C}h^{2k}\|\omega\|_{1+k}^2+\frac{\alpha}{2}\left\|\nabla_{d,r}e\right\|^2. \tag{28}$$

From (26) and (28), we get I^2 as follows:

$$\mathrm{I}^2=\mathrm{C}h^{2k+2}\|\omega\|_{1+k}^2-\mathrm{C}h^{2k}\|\omega\|_{1+k}^2. \tag{29}$$

To find I^3,

$$\begin{aligned} \left|\mathrm{I}^3\right| &\leq \frac{1}{2\emptyset}\left\|\pi_h(\omega)-\left(Q^0\omega\right)\right\|^2+\frac{\|\beta\|^2\emptyset}{2}\left\|\nabla_{d,r}e\right\|^2 \\ &\leq ch^{2k+2}\|\omega\|_{1+k}^2+\frac{c\|\beta\|^2\emptyset}{2}\|\,e\|^2 \\ &\leq Ch^{2k+2}\|\omega\|_{1+k}^2+C\|e\|^2. \end{aligned} \tag{30}$$

To estimate I^4 by Cauchy-Schwartz and Young's-inequality and Lemma 6.1, we obtain

$$\begin{aligned} \left|\mathrm{I}^4\right| &\leq \frac{\|c\|^2}{2\vartheta}\left\|\omega-\left(Q^0\omega\right)\right\|^2+\frac{\vartheta}{2}\|e\|^2, \\ \left|\mathrm{I}^4\right| &\leq Ch^{2k}\|\omega\|_{1+k}^2+\frac{\vartheta}{2}\|e\|^2. \end{aligned} \tag{31}$$

Substituting (25), (29), (30), (31) into (24), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|e\|^2+\frac{\vartheta}{2}\|e\|^2+\frac{\vartheta}{2}\left\|\nabla_{d,r}e\right\|^2 &\leq Ch^{2k+2}\|\omega\|_{1+k}^2+Ch^{2k}\|\omega\|_{1+k}^2-Ch^{2k}\|\omega\|_{1+k}^2+\frac{\vartheta}{2}\|e\|^2, \\ \frac{1}{2}\frac{d}{dt}\|e\|^2+\frac{\vartheta}{2}\|e\|^2+\frac{C\vartheta}{2}\|e\|^2 &\leq Ch^{2k+2}\|\omega\|_{1+k}^2+\frac{\vartheta}{2}\|e\|^2. \end{aligned}$$

Multiply by 2 and integral both side from $(0$ to $t)$, we obtain

$$\|e\|^2+C_1\int_0^t\|e(\tau)\|^2d\tau \leq Ch^{2k+2}\int_0^t\|\omega(\tau)\|_{1+k}^2d\tau+\|e(0)\|^2, \tag{32}$$

where $C_1=\vartheta C$. □

7 Error Analysis in H^1 Norm

Starting with the next lemma, we provide an estimate for the difference between the SLD-WGFEM approximation, represented as ω_h, and the L^2 projection of the exact solution for the original problem.

Lemma 7.1. *Let ω be the solution of (1) and $\omega_h\in S_h(m,l)$ be the SLDWGFEM approximation of ω arising from (4). Denote by $e_h=\omega_h-Q_h\omega$ the difference between the SLDWGFEM approximation and the L^2 projection of the exact solution $\omega=(\omega_1,\omega_2)$. Then, there exists a constant C such that*

$$\|e^0(t)\|^2+\eta\|\nabla_{d,r}e(t)\|^2 \leq \int_0^t Ch^{2k}\|\omega_h(\tau)\|^2d\tau+(\|e^0(0)\|^2+\eta\|\nabla_{d,r}e(0)\|^2).$$

Proof. Substituting v in inequality (23) by $e = e_t$, we arrive at

$$\begin{aligned}\left(e_t^0, e_t^0\right) + a_{SLDWG}(e_t, e_t) = &\left(\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon Q(\nabla_{d,r}\omega), \nabla_{d,r}e_t\right) + \left(c\omega - c\left(Q^0\omega\right), e_t^0\right)\\ &+ \delta\left(\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right) - \beta\cdot Q(\nabla\omega), \beta\cdot\nabla_{d,r}e_t\right)\\ &+ \left(\pi_h(\beta\cdot\nabla_{d,r}\omega) - (\beta\cdot Q\nabla_{d,r}\omega), e_t^0\right).\end{aligned} \tag{33}$$

Rewrite (33), we have

$$\begin{aligned}\left(e_t^0, e_t^0\right) + a_{SLDWG}(e_t, e_t) = &\left(\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon Q(\nabla_{d,r}\omega), \nabla_{d,r}e_t\right) + \left(c\omega - c\left(Q^0\omega\right), e_t^0\right)\\ &+ \delta\left(\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right) - \beta\cdot Q(\nabla\omega), \beta\cdot\nabla_{d,r}e_t\right)\\ &- \left(\pi_h(\omega) - (Q\omega), \beta\nabla_{d,r}e_t^0\right).\end{aligned} \tag{34}$$

Using Lemma 4.1 to $a_{SLDWG}(e_t, e_t)$, we have

$$\left\|e_t^0\right\|^2 + \frac{\emptyset}{2}\left\|e_t^0\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2 \le R_1 + R_2 + R_3 - R_4. \tag{35}$$

Using Lemma 6.1 and Lemma 6.2 for $|R_i, i = 1,2,3,4|$, we have

$$\begin{aligned}|R_1| &= \left|\left(\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon Q_h(\nabla_{d,r}\omega), \nabla_{d,r}e_t\right)\right|\\ &\le \frac{1}{2\emptyset}\left\|\pi_h\left(\epsilon\nabla_{d,r}\omega\right) - \epsilon Q_h(\nabla\omega)\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2\\ &\le Ch^{2k}\|\omega_h\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2,\\ |R_2| &= \delta\left|\left(\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right) - \beta\cdot Q_h(\nabla\omega), \beta\cdot\nabla_{d,r}e_t\right)\right|\\ &\le \frac{\delta^2}{2\emptyset_1}\left\|\pi_h\left(\beta\cdot\nabla_{d,r}\omega\right) - \beta\cdot Q_h(\nabla\omega)\right\|^2 + \frac{\emptyset_1}{2}\left\|\beta\cdot\nabla_{d,r}e_t\right\|^2\\ &\le \frac{(h-\epsilon)\|\beta\|^2}{2\emptyset_1}\left\|\pi_h\left(\nabla_{d,r}\omega\right) - Q_h(\nabla\omega)\right\|^2 + \frac{\emptyset_1\|\beta\|^2}{2}\left\|\nabla_{d,r}e_t\right\|^2\\ &\le \mathrm{C}h^{2k+2}\left\|\omega_h\right\|^2 - \mathrm{C}h^{2k}\left\|\omega_h\right\|^2 + \emptyset_2\left\|\nabla_{d,r}e_t\right\|^2\\ &\le \mathrm{C}h^{2k+2}\left\|\omega_h\right\|^2 - \mathrm{C}h^{2k}\left\|\omega_h\right\|^2 + \emptyset_2\left\|\nabla_{d,r}e_t\right\|^2,\\ |R_3| &= \left(c\omega - c\left(Q^0\omega\right), e_t^0\right)\\ &\le \mathrm{C}h^{2k}\left\|\omega_h^0\right\|^2 + \frac{\emptyset}{2}\|e_t^0\|^2,\\ |R_4| &= \left(\beta\pi_h(\omega) - (\beta Q\omega), \nabla_{d,r}e_t\right)\\ &\le \frac{\|\beta\|^2}{2\emptyset}\left\|\pi_h(\omega) - Q_h(\omega)\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2\\ &\le \frac{\|\beta\|^2}{2\emptyset}\left\|\pi_h(\omega) - Q_h(\omega)\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2\\ &\le \mathrm{C}h^{2k+2}\left\|\omega_h\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2.\end{aligned} \tag{36}$$

Substituting $|R_1|, |R_2|, |R_3|, |R_4|$ from (36) into inequality (35), we get

$$\begin{aligned}\left\|e_t^0\right\|^2 + \frac{\emptyset}{2}\left\|e_t^0\right\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2 \leq Ch^{2k}\|\omega_h\|^2 + \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t\right\|^2 + \mathrm{C}h^{2k+2}\left\|\omega_h\right\|^2 - \mathrm{C}h^{2k}\left\|\omega_h\right\|^2 \\ + \emptyset_2\left\|\nabla_{d,r}e_t\right\|^2 + \mathrm{C}_2h^{2k}\left\|\omega_h^0\right\|^2 + \frac{\emptyset}{2}\|e_t^0\|^2 - \mathrm{C}h^{2k+2}\left\|\omega_h\right\|^2 \\ - \frac{\emptyset}{2}\left\|\nabla_{d,r}e_t^0\right\|^2, \\ \left\|e_t^0\right\|^2 + \eta\left\|\nabla_{d,r}e_t\right\|^2 \leq Ch^{2k}\|\omega_h\|^2, \\ \frac{d}{dt}(\left\|e^0\right\|^2 + \eta\left\|\nabla_{d,r}e\right\|^2) \leq Ch^{2k}\|\omega_h\|^2,\end{aligned}$$

where $\eta = \frac{\emptyset}{2} - \emptyset_2$, and integrate the inequality from 0 to t to accumulate the error effect over time:

$$\int_0^t \frac{d}{d\tau}\left(\|e^0(\tau)\|^2 + \eta\|\nabla_{d,r}e(\tau)\|^2\right)d\tau \leq \int_0^t Ch^{2k}\|\omega_h(\tau)\|^2 d\tau.$$

Simplify the integrated inequality to:

$$\|e^0(t)\|^2 + \eta\|\nabla_{d,r}e(t)\|^2 \leq \int_0^t Ch^{2k}\|\omega_h(\tau)\|^2 d\tau + (\|e^0(0)\|^2 + \eta\|\nabla_{d,r}e(0)\|^2).$$

□

8 Numerical Results

We introduce the linear convection-diffusion problem (1) with homogeneous Dirichlet boundary condition and initial condition. On $\Omega = [0,1]\times[0,1]$, and let $\epsilon = 0.0001$ and time range be $(0,T) = (0,1], \mathrm{c} = 1$, and the calculation of the velocity vector will be as follows:

$$\beta = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right),$$

the initial and boundary conditions and the source term $f(x,t)$ can be obtained from the example solution. The exact solution [7] is chosen in the example

$$\omega = e^{-t}\sin(\pi x)\sin(\pi y).$$

To begin, we divide the square domain $\Omega = [0,1]\times[0,1]$ into $\mathrm{N}\times\mathrm{N}$ sub-squares of equal size. Next, we partition each square element into two triangles using a diagonal line with a negative slope, thereby creating a triangular mesh. Let $h = 1/_N (N = 2,4,16,32,64)$ represent the spatial mesh size. We select a sufficiently small time step $\mathcal{T} = 0.001$ and calculate the error of $\omega - \omega_h$ for L^2 and H^1 norms, considering the mesh size. This analysis further validates the theoretical findings. The obtained results are as follows:

- The Table 1 and Figure 1 give the WGFEM numerical solution and exact solution for the mesh size $h = \frac{1}{64}$.
- The Table 2 and Figure 2 give the results for the CADWGFEM was introduced.
- The Table 3 and Figure 3 give the results for the SLDWGFEM.

Table 1: H^1- error and L^2- error for WGFEM with $\mathcal{T} = 10^{-3}$, $\epsilon = 10^{-4}$.

h	L^2- error	L^2- order	H^1- error	H^1- order
$2.5000e-01$	$8.1892e-01$	0	$9.3962e-01$	0
$1.2500e-01$	$2.0836e-01$	$1.9746e+00$	$4.6035e-01$	$1.0293e+00$
$6.2500e-02$	$5.2079e-02$	$2.0003e+00$	$2.2869e-01$	$1.0093e+00$
$3.1250e-02$	$1.2995e-02$	$2.0027e+00$	$1.1403e-01$	$1.0040e+00$
$1.5625e-02$	$3.2353e-03$	$2.0060e+00$	$5.6929e-02$	$1.0021e+00$

Table 2: H^1- error and L^2- error for CADWGFEM with $\mathcal{T} = 10^{-3}$, $\epsilon = 10^{-4}$.

h	L^2- error	L^2- order	H^1- error	H^1- order
2.5000e − 01	1.6378e − 02	0	3.1321e − 02	0
1.2500e − 01	4.1672e − 03	1.9746e + 00	1.5345e − 02	1.0293e + 00
6.2500e − 02	1.0416e − 03	2.0003e + 00	7.6232e − 03	1.0093e + 00
3.1250e − 02	2.5990e − 04	2.0027e + 00	3.8009e − 03	1.0040e + 00
1.5625e − 02	6.4707e − 05	2.0060e + 00	1.8976e − 03	1.0021e + 00

Table 3: H^1- error and L^2- error for SLDWGFEM with $\mathcal{T} = 10^{-3}$, $\epsilon = 10^{-4}$.

h	L^2- error	L^2- order	H^1- error	H^1- order
2.5000e − 01	5.5829e − 03	0	7.8587e − 03	0
1.2500e − 01	1.4346e − 03	1.9604e + 00	3.6161e − 03	1.1199e + 00
6.2500e − 02	3.5821e − 04	2.0018e + 00	1.7581e − 03	1.0404e + 00
3.1250e − 02	8.9248e − 05	2.0049e + 00	8.6948e − 04	1.0158e + 00
1.5625e − 02	2.2278e − 05	2.0022e + 00	4.3331e − 04	1.0047e + 00

Then, the figure is as follows:

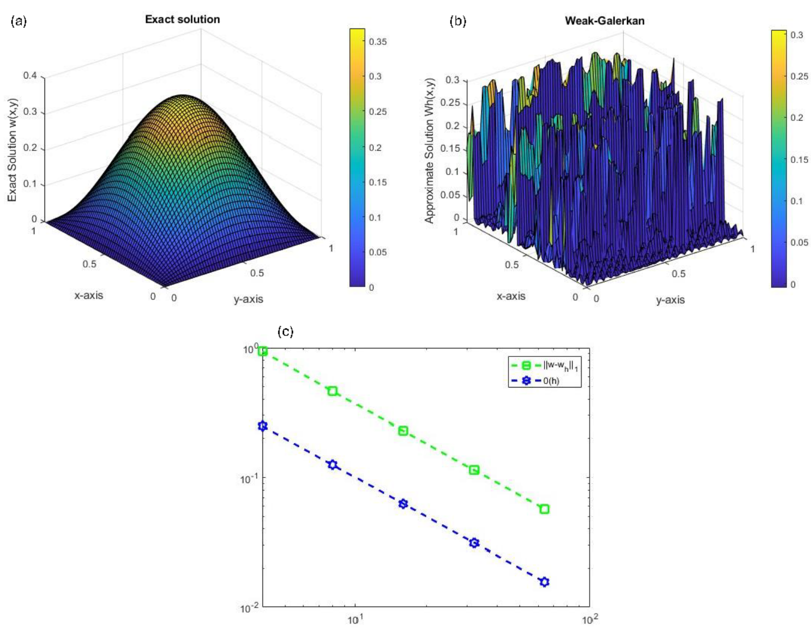

Figure 1: (**a**) The exact solution for problem (1). (**b**) The approximate solution for problem (1) by WGFEM. (**c**) Error and order error for H^1-norm.

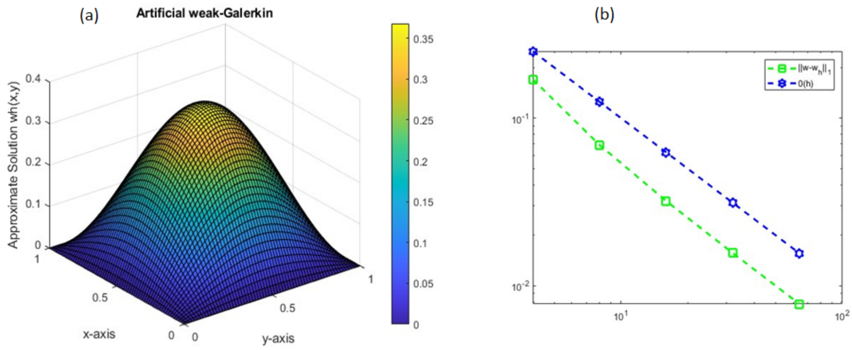

Figure 2: (**a**) The approximate solution for problem (1) by CADWGFEM. (**b**) Error and order error for H^1-norm.

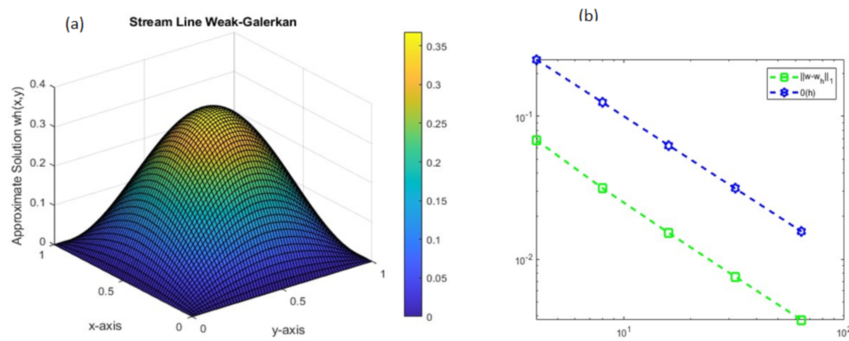

Figure 3: (**a**) The approximate solution for problem (1) by SLDWGFEM. (**b**) Error and order error for H^1-norm.

The SLDWGFEM approximation (4) can be represented as a matrix problem for each element, the local stiffness matrix is calculated, making the calculation of the element stiffness matrix an essential step in implementing the method on a computer. For each triangular element $\aleph$, there exists a collection of basis functions denoted as $P_m(\aleph)$. These basis functions are represented by $\lambda^{0,i}$, where i varies from 1 to M^0. Additionally, there is a basis function set for all edges of the element, denoted by $\lambda^{b,i}$, where i varies from 1 to M^b. The set $P_m(\aleph)$ represents a group of basis functions for $\sum_{e\in\partial\aleph\cap\partial\Lambda} P_\ell(\partial\aleph)$.

Every weak function $\rho_h = \left\{\rho^0, \rho^b\right\} \in \mathbb{S}_h(m,\ell)$ can be represented in this way:

$$\rho_h|_T = \left\{\sum_{i=1}^{N^0} \rho^{0,m}\lambda^{0,m}, \sum_{i=1}^{N^b} \rho^{b_l m}\lambda^{b,m}\right\}\left\{\overline{\lambda^0}\cdot\overline{\rho^0}, \overline{\lambda^b}\cdot\overline{\rho^b}\right\},$$

$$\overline{\rho^0} = \begin{pmatrix} \rho^{0,1} \\ \rho^{0,2} \\ \vdots \\ \rho^{0,M^0} \end{pmatrix}, \quad \overline{\rho^b} = \begin{pmatrix} \rho^{b,1} \\ \rho^{b,2} \\ \vdots \\ \rho^{b,M^b} \end{pmatrix}.$$

The SLDWGFEM approximation (4) on $\aleph$ is expressed as follows: For $j = 1, 2, \ldots, M$, find $(w_h^j) \in \mathbb{S}_h^0$ and

$$\left(\delta_t w_h^j, \rho^0\right) + a\left(w_h^j, \rho^0\right) = \left(f^j, \rho^0\right), \quad \forall\rho \in \mathbb{S}_h^0, \tag{37}$$

$$\delta_t w_h^j = \frac{w^{j+1} - w^j}{\Delta t}, \quad \Delta t = t_{n+1} - t_n.$$

The local matrix form of (37) is

$$\mu_1 \left(W^{j+1} - W^j\right) + \Delta t \left(\mu_2 - \mu_3 + \mu_4 + \mu_5\right) W^j = \Delta t F^j, \tag{38}$$

where

$$\mu_i = \begin{bmatrix} \mu_{0,0}^i & \mu_{b,0}^i \\ \mu_{0,b}^i & \mu_{b,b}^i \end{bmatrix}, \quad W^j = \begin{bmatrix} \frac{w_0^j}{w_b^j} \end{bmatrix}, \quad F^j = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M_0} \end{bmatrix}, \tag{39}$$

where $\mu_{0,0}^i$ is an $M_0 \times M_0$ matrix, $\mu_{0,b}^i$ is an $M_0 \times M_b$ matrix, $\mu_{b,0}^i$ is an $M_b \times M_0$ matrix, $\mu_{b,b}^i$ is an $M_b \times M_b$ matrix, and i ranges from 1 to 4. The matrix definitions are as follows:

$$\begin{aligned} \mu_{0,0}^1 &= \left((\Phi_{0,l}, \Phi_{0,i})|_{\aleph}\right)_{i,l}, \quad \mu_{0,b}^1 = \left((\Phi_{b,l}, \Phi_{0,i})|_{\aleph}\right)_{i,l}, \\ \mu_{b,0}^1 &= \left((\Phi_{0,l}, \Phi_{b,i})|_{\aleph}\right)_{i,l}, \quad \mu_{b,b}^1 = \left((\Phi_{b,l}, \Phi_{b,i})|_{\aleph}\right)_{i,l}, \end{aligned} \qquad \text{(Mass matrix)},$$

$$\begin{aligned} \mu_{0,0}^2 &= \left((\nabla_r\Phi_{0,l}, \nabla_r\Phi_{0,i})|_{\aleph}\right)_{i,l}, \quad \mu_{0,b}^2 = \left((\nabla_r\Phi_{b,l}, \nabla_r\Phi_{0,i})|_{\aleph}\right)_{i,l}, \\ \mu_{b,0}^2 &= \left((\nabla_r\Phi_{0,l}, \nabla_r\Phi_{b,i})|_{\aleph}\right)_{i,l}, \quad \mu_{b,b}^2 = \left((\nabla_r\Phi_{b,l}, \nabla_r\Phi_{b,i})|_{\aleph}\right)_{i,l}, \end{aligned} \qquad \text{(Diffusion matrix)},$$

$$\begin{aligned} \mu_{0,0}^3 &= \left((\nabla_r\Phi_{0,l}, \Phi_{0,i})|_{\aleph}\right)_{i,l}, \quad \mu_{0,b}^3 = \left((\nabla_r\Phi_{b,l}, \Phi_{0,l})|_{\aleph}\right)_{i,l}, \\ \mu_{b,0}^3 &= \left((\nabla_r\Phi_{0,l}, \Phi_{b,i})|_{\aleph}\right)_{i,l}, \quad \mu_{b,b}^3 = \left((\nabla_r\Phi_{b,l}, \Phi_{b,i})|_{\aleph}\right)_{i,l}, \end{aligned} \qquad \text{(Convection matrix)},$$

where l and i stand for the column and row matrices, respectively. Each element of the vector f_j is defined as $f_j = \int_{\aleph} f \overline{\rho^b} dz$. This vector is represented as $F^{\mathrm{j}} = (f_j)$.

To determine the matrices $\mu_i, i = 1, 2, 3, 4$, we need to calculate the discrete gradient operator $\nabla_d A$ for the weak function $\rho_h|_{\aleph}$, which is expressed as a locally vector. Let $\eta_i, i = 1, 2, \ldots, M^v$ be a collection of basis functions of $\mathbb{V}_r(\aleph)$. For any $q_h \in \mathbb{S}_h$, it can then be represented as follows:

$$q_h|_{\aleph} = \sum_{i=1}^{M^v} q^i \eta_i,$$

where

$$\bar{q} = \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^{M^v} \end{pmatrix}.$$

Based on the definition of the discrete weak gradient operator, the vector $\nabla_r \rho_h$ can be expressed as follows:

$$D_{\aleph}\left(\overline{\nabla_r \rho_h}\right) = -Z_{\aleph}\overline{\rho^0} + T_{\aleph}\overline{\rho^b}, \tag{40}$$

where $[D_\aleph]_{M_v\times M_v}$, $[Z_\aleph]_{M_v\times M_0}$ and $[T_\aleph]_{M_v\times M_b}$ are defined as follows:

$$D_\aleph = \begin{pmatrix} \int_\aleph \eta_1\cdot\eta_1\, d\Lambda & \cdots & \int_\aleph \eta_1\cdot\eta_{M_v}\, d\Lambda \\ \vdots & \ddots & \vdots \\ \int_\aleph \eta_{M_v}\cdot\eta_1\, d\Lambda & \cdots & \int_\aleph \eta_{M_v}\cdot\eta_{M_v}\, d\Lambda \end{pmatrix},$$

$$Z_\aleph = \begin{pmatrix} \int_\aleph (\nabla\cdot\eta_1)\,\Phi_{0,1}\, d\Lambda & \cdots & \int_\aleph (\nabla\cdot\eta_1)\,\Phi_{0,M_0}\, d\Lambda \\ \vdots & \ddots & \vdots \\ \int_\aleph (\nabla\cdot\eta_{M_v})\,\Phi_{0,1}\, d\Lambda & \cdots & \int_\aleph (\nabla\cdot\eta_{M_v})\,\Phi_{0,M_v}\, d\Lambda \end{pmatrix},$$

$$T_\aleph = \begin{pmatrix} \int_{\partial\aleph} (\eta_1\cdot n)\,\Phi_{b,1}\, ds & \cdots & \int_{\partial\aleph} (\eta_1\cdot n)\,\Phi_{b,M_b}\, ds \\ \vdots & \ddots & \vdots \\ \int_{\partial\aleph} (\eta_{M_0}\cdot n)\,\Phi_{b,1}\, ds & \cdots & \int_{\partial\aleph} (\eta_{M_v}\cdot n)\,\Phi_{b,M_b}\, ds \end{pmatrix}.$$

After computing the matrices $D_\aleph$, $Z_\aleph$, and $T_\aleph$, one can use (40) to compute the weak gradient of basis functions $\Phi_{0,j}$ and $\Phi_{b,J}$, as shown in [28].

$$\left(\overline{\nabla_r\Phi_{0,J}}\right) = -D_\aleph^{-1}Z_\aleph e_J^{-N_o},\ \left(\overline{\nabla_r\Phi_{b,J}}\right) = D_\aleph^{-1}T_\aleph e_J^{-M_b}.$$

Here, $e_J^{-M_0}$ and $e_J^{-M_b}$ are the standard basis vectors. Let us define:

$$\begin{aligned} (Y_\aleph)_{M_v\times M_v} &= [h\,(\eta_\ell,\eta_J)_\aleph]_{J,\ell}\,, \\ (U_\aleph)_{M_0,\times M_v} &= \left[(\beta.\eta_\ell,\Phi_{0,J})_\aleph\right]_{J,\ell}\,, \\ (N_\aleph)_{M_v\times M_v} &= [(\beta\cdot\eta_\ell,\beta\cdot\eta_J)_\aleph]_{J,\ell}\,, \\ (K_\aleph)_{M_0,\times M_0} &= \left[(\mathrm{c}\Phi_{0,\ell},\Phi_{0,J})_\aleph\right]_{J,\ell}\,. \end{aligned}$$

The elementary matrix can represent the local stiffness matrix by substituting the values M_i, $i = 1,2,3,4,5$ into (38). A matrix problem can now be formulated using the SLDWGFEM, which is described by the following lemma.

Lemma 8.1. *The local stiffness matrix of SLDWGFEM for* (38) *is* $\mu_i, i = 1,2,3,4,5$ *can be expressed as*

$$\begin{aligned} \mu_1 &= \begin{pmatrix} K_\aleph & 0 \\ 0 & 0 \end{pmatrix}, \\ \mu_2 &= \begin{pmatrix} Z_\aleph^{\hat t}D_\aleph^{-\hat t}Y_\aleph D_\aleph^{-1}Z_\aleph & -Z_\aleph^{\hat t}D_\aleph^{\hat t}Y_\aleph D_\aleph^{-1}T_\aleph \\ -T_\aleph^{\hat t}D_\aleph^{-\hat t}Y_\aleph D_\aleph^{-1}Z_\aleph & T_\aleph^{\hat t}D_\aleph^{-\hat t}Y_\aleph D_\aleph^{-1}T_\aleph \end{pmatrix}, \\ \mu_3 &= \begin{pmatrix} Z_\aleph^{\hat t}D_\aleph^{-\hat t}N_\aleph D_\aleph^{-1}Z_\aleph & -Z_\aleph^{\hat t}D_\aleph^{-\hat t}N_\aleph D_\aleph^{-1}T_\aleph \\ -T_\aleph^{\hat t}D_\aleph^{-\hat t}N_\aleph D_\aleph^{-1}Z_\aleph & T_\aleph^{\hat t}D_\aleph^{-\hat t}N_\aleph D_\aleph^{-1}T_\aleph \end{pmatrix}, \\ \mu_4 &= \begin{pmatrix} -U_\aleph D_\aleph^{-1}Z_\aleph & U_\aleph D_\aleph^{-1}T_\aleph \\ T_\aleph^{\hat t}D_\aleph^{-\hat t}U_\aleph^{\hat t} & 0 \end{pmatrix}, \end{aligned} \tag{41}$$

where $\hat t$ *stands for the standard matrix transpose and* $\mu_1 = \mu_5$.

9 Conclusions

This paper presents and analyses a new numerical scheme for solving elliptic partial differential equations, the SLDWGFEM. By addressing the inherent limitations of conventional WGFEM and its variant, CADWGFEM, in particular with respect to turbulence and stability issues associated with small turbulence parameters (ϵ) compared to the grid size (h), our study makes a significant methodological advance. The introduction of a positive parameter (δ) allows to simplify the propagation, which improves the accuracy of the solution, while reducing unnecessary crosswind smoothing in CADWGFEM. In addition, it reduces noticeable oscillations, which is a common problem in WGFEM. This approach provides a better balance between stability and accuracy, significantly improving the performance of previous methods. Our results highlight the importance of choosing a suitable diffusion direction and magnitude. This is demonstrated by the excellent handling of crosswind propagation and the stability of the SLDWGFEM without affecting the accuracy of the solution. Error estimation and optimal ordering in the discrete $\mathbb{L}^2$-norm and $\mathbb{H}^1$-norm further validate the theoretical underpinnings of our method, with numerical results reinforcing its effectiveness.

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